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Connections Among Quantum Logics. Part 1. Quantum Propositional Logics

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In this paper, we propose a theory of quantum logics which is general enough to enable us to reexamine previous work on quantum logics in the context of this theory. It is then easy to assess the differences between the different systems studied. The quantum logical systems which we incorporate are divided into two groups which we call "quantum propositional logics" and "quantum event logics." We include the work of Kochen and Specker (partial Boolean algebras), Greechie and Gudder (orthomodular partially ordered sets), Domotar (quantum mechanical systems), and Foulis and Randall (operational logics) in quantum propositional logics; and Abbott (semi-Boolean algebras) and Foulis and Randall (manuals) in quantum event logics. In this part of the paper, we develop an axiom system for quantum propositional logics and examine the above structures in the context of this system.

1. INTRODUCTION

The general term *quantum logic* has been used to refer to a large assortment of mathematical systems, and it is natural to ask how all these structures compare. In order to determine which of these structures are generalizations of other structures, or, in fact, which are equivalent, we introduce a theory which is general enough to encompass all of these mathematical systems. We may then examine partial Boolean algebras (see the work of Kochen and Specker, 1967), orthomodular partially ordered sets (see Greechie and Gudder, 1973), quantum mechanical systems (see Domotar, 1974), semi-Boolean algebras (see Abbott, 1967), and manuals and their associated operational logics (see Foulis and Randall, 1979) in the context of this theory, as well as investigating all the intermediate systems.

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The current work on quantum logics is divided into two distinct groups, which we call here quantum event logics and quantum propositional logics. The work of Abbott on semi-Boolean algebras and Foulis and Randall on manuals falls into the general category of quantum event logics. The other theories mentioned above fall into the category of quantum propositional logics. Quantum event logics lie "under" quantum propositional logics in the sense that the latter are shown to be the former modulo an equivalence relation. This connection between the two approaches, in the specific case of manuals and operational logics, has been studied for some time by Foulis and Randall. More will be said about quantum event logics and this connection in the second part of this paper. In addition, many of the examples and counterexamples mentioned here will be included instead in Part II of this paper, as they are derived from the work on manuals which will be discussed there.

Most of the work that has been done on quantum logics has been in the area of quantum propositional logics, and in this part of the paper we will investigate that work. There are two distinct approaches that have been used in this area: one may begin with a set of elements on which is defined an orthogonality relation, or one may begin with a set of Boolean algebras which are pasted together in some way. Both approaches are considered here, and we show the connections between the two.

The reader is referred to the above-mentioned references for the motivation and origins of each of the individual structures listed above. Here, we will begin with these structures and develop an underlying theory which will enable us to compare them. We conclude at the end of the paper with a diagram which includes all these structures and illustrates the connections between them.

2. ORTHO-ALGEBRAS

We define here a structure which is sufficiently general to subsume all previously mentioned quantum propositional logics, but which is at the same time sufficiently small to be interesting. The system we propose is what we call an ortho-algebra.

Definition. An *ortho-algebra* is a set L, a partial binary relation $\perp$ on the set L, a partial operation $\oplus: \mathrm{L} \times \mathrm{L} \to \mathrm{L}$ such that $a \oplus b$ is defined if and only if $a \perp b$, a map $': L \to L$ denoted $a \to a'$ for all $a \in L$, and two elements $0, 1 \in L$, such that for all $a, b \in L$, the following are satisfied:

(i) If $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$;

(ii) $a \perp 0$ and $a \oplus 0 = a$;

(iii) $a \perp a'$ and $a \oplus a' = 1$;

(iv) if $a \perp (a' \oplus b)$, then $b = 0$;
(v) if $a \perp (a \oplus b)$, then $a = 0$;
(vi) if $a \perp b$, then $a \perp (a \oplus b)'$ and $b' = a \oplus (a \oplus b)'$.

From this, we easily see that the following facts are true:

Proposition. Let L be an ortho-algebra, $a, b \in L$. Then (a) $0' = 1$, $1' = 0$; (b) $(a')' = a$; (c) if $a \oplus b = a \oplus c$, then $b = c$; (d) if $a \oplus b = 1$, then $b = a'$.

In an ortho-algebra, we do not require that the orthogonal sum operation be associative. We say that an ortho-algebra L is *associative* if, for all a, b, c in L, the following condition is satisfied: if $a \perp b$ and $c \perp (a \oplus b)$, then $b \perp c$ and $a \perp (b \oplus c)$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

Definition. Let L be an ortho-algebra with $a, b \in L$. We say $a \leq b$ if and only if there is a $c \in L$ with $a \perp c$ and $a \oplus c = b$.

It is easily shown that the relation $\leq$ is reflexive and antisymmetric. In general, though, it will not be transitive. One can show, however, that under the additional constraint of associativity, $\leq$ is transitive and hence a partial ordering. It is possible for the relation $\leq$ to be transitive on an ortho-algebra which is not associative. Hence the class of ortho-algebras on which $\leq$ is transitive constitutes an intermediate system between ortho-algebras and associative ortho-algebras. The following facts will be useful later on, and are easily seen to be true:

Proposition. Let a, b be elements in an ortho-algebra L. Then (1) $a \leq b$ if and only if $b' \leq a'$; (2) $a \leq b$ if and only if $a \perp b'$; (3) if L is associative and $a \perp b$, then $x \leq a, b$ implies $x = 0$.

3. BOOLEAN ATLASES

We now consider the second approach to quantum propositional logics, and begin with a family of Boolean algebras. The most general structure here is that of a Boolean atlas, as defined below. We will show that Boolean atlases are equivalent to ortho-algebras. We examine Boolean atlases now for two reasons: much of the work done on quantum propositional logics is presented from this viewpoint, and many of the concepts we will define on an ortho-algebra are more intuitively clear in the context of Boolean atlases.

Definition. A family $(B_i : i \in I)$ of Boolean algebras is called a *Boolean atlas* if it satisfies the following conditions (here, we use the notation $\leq_i$, 0_i, 1_i, C_i, M_i, J_i to refer to the order relation, the 0 and 1 elements, the complement, the meet, and the join, respectively, in a Boolean algebra B_i):

(i) If $B_i \subseteq B_j$, then $B_i = B_j$;
(ii) if $a, b \in B_i \cap B_j$, then $a \leq_i b$ if and only if $a \leq_j b$;
(iii) $1_i = 1_j$ and $0_i = 0_j$ for all $i, j \in I$;
(iv) if $a \in B_i \cap B_j$, then $C_i(a) = C_j(a)$;
(v) if $a, b \in B_i \cap B_j$ and if either $M_i(a, b) = 0$ or $M_j(a, b) = 0$, then $M_i(a, b) = M_j(a, b)$ and $J_i(a, b) = J_j(a, b)$.

Notice that we may have $a, b \in B_i \cap B_j$ and yet $M_i(a, b) \neq M_j(a, b)$ and $J_i(a, b) \neq J_j(a, b)$. We define a *Boolean manifold* to be a Boolean atlas which satisfies the condition: if $a, b \in B_i \cap B_j$, then $M_i(a, b) = M_j(a, b)$ and $J_i(a, b) = J_j(a, b)$.

We will need the following definitions:

Definitions. Let $\mathscr{B} = (B_i \colon i \in I)$ be a Boolean atlas, $a, b \in \bigcup B_i$ and $S \subseteq \bigcup B_i$. Then we say the following:

(a) a and b are *orthogonal* if there is an $i \in I$ with $a, b \in B_i$ and $M_i(a, b) = 0$; a subset S is called *pairwise orthogonal* if a and b are orthogonal for all a, b in S.

(b) a and b are *compatible* if there is an $i \in I$ with $a, b \in B_i$; a subset S is called *pairwise compatible* if a and b are compatible for all a, b in S.

(c) S is *jointly compatible* if there is an $i \in I$ with $S \subseteq B_i$.

(d) S is *jointly orthogonal* if there is an $i \in I$ with $S \subseteq B_i$ and S is pairwise orthogonal.

We will return to Boolean atlases later. First, we must develop additional structure on ortho-algebras.

4. ADDITIONAL CONDITIONS ON ORTHO-ALGEBRAS

A subset A of an ortho-algebra L is called pairwise orthogonal if $a \perp b$ for all $a, b \in A$. We wish to define the notion of joint orthogonality on finite subsets. Intuitively, we will say a set $A = \{a_1, a_2 \ldots, a_n\}$ is jointly orthogonal if the orthogonal sum $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ exists (that is, the sums are defined for any rearrangement and any placing of parentheses). To make this notation precise, let A be a finite subset of L with n elements. We use induction on n to define orthogonality of A, and to define the orthogonal sum of A, denoted $\sum A$, which is defined if and only if A is jointly orthogonal. If $n = 2$, say $A = \{a, b\}$, we say A is jointly orthogonal if $a \perp b$, in which case $\sum A = a \oplus b$. If $\# A = n$, we say A is *jointly orthogonal* if the following three conditions are satisfied:

(i) For all $a \in A$, $A - \{a\}$ is jointly orthogonal.
(ii) For all $a \in A$, $a \perp (A - \{a\})$.
(iii) $a \oplus (A - \{a\}) = b \oplus (A - \{b\})$ for all a, b in A.

In this case, we define $\sum A = a \oplus (A - \{a\})$ for any a in A. Clearly, if a subset A is jointly orthogonal then it is pairwise orthogonal. However, the converse is not true.

Let L be an ortho-algebra, A a subset of L with a, b in A. Then the pair a, b is said to have a *Mackey decomposition* in A if there exists a jointly orthogonal triple $\{a_0, b_0, c\}$ in A such that $a = a_0 \oplus c$ and $b = b_0 \oplus c$. If the triple is unique, the pair a, b is said to have a unique Mackey decomposition in A. We say that a, b in L are *compatible* if they have a Mackey decomposition in L, and we say they are *uniquely compatible* if they have a unique Mackey decomposition in L. It is possible for two elements to be compatible yet not uniquely compatible. An ortho-algebra L is said to have the *unique Mackey decomposition* (UMD) *property* if every compatible pair is uniquely compatible.

Let A be a finite subset of an ortho-algebra L. We say that A is *jointly compatible* if there is a (finite) jointly orthogonal subset T of L such that $A \subseteq \{\sum T_0 \colon T_0 \subseteq T\}$. It is easy to see that if A is jointly compatible then it is pairwise compatible: again, however, the converse is not true. We may now define the two notions of coherence on an ortho-algebra:

Definitions. (1) An ortho-algebra is called *orthocoherent* if every finite subset that is pairwise orthogonal is jointly orthogonal.

(2) An ortho-algebra is called *compatibly coherent* if every finite subset that is pairwise compatible is jointly compatible.

It is easy to see that if an ortho-algebra is compatibly coherent then it is orthocoherent. An example of an ortho-algebra which is orthocoherent but not compatibly coherent may be found in Pool (see Pool, 1963). It can be shown that if L is an orthocoherent ortho-algebra, then L is transitive if and only if L is associative.

Theorem. Let L be an associative ortho-algebra. If L is orthocoherent, then L satisfies the unique Mackey decomposition property.

Proof. Let a, b be compatible elements, and suppose $\{a_0, b_0, c\}$ and $\{a_1, b_1, d\}$ are two Mackey decompositions for the pair a, b. Let $x = a_0 \oplus b_0 \oplus c$. Since $d \leq a$ and $a \leq x$, by associativity (and hence transitivity) of L, we have $d \leq x$. Similarly, $a_1 \leq x$ and $b_1 \leq x$. Therefore, $d \perp x'$, $a_1 \perp x'$, and $b_1 \perp x'$. By orthocoherence, we have $(a_1 \oplus b_1 \oplus d) \perp x'$, and hence $(a_1 \oplus b_1 \oplus d) \leq x = (a_0 \oplus b_0 \oplus c)$. By symmetry, then, we have $a_1 \oplus b_1 \oplus d = a_0 \oplus b_0 \oplus c$. Since $a_1 \oplus d \oplus b_1 = a \oplus b_1 = a_0 \oplus c \oplus b_1$, and $a_0 \oplus c \oplus b_0 = a_0 \oplus b = a_0 \oplus d \oplus b_1$, we have $(a_0 \oplus b_1) \oplus c \subseteq = (a_0 \oplus b_1) \oplus d$ and hence $c = d$. It then follows immediately that $a_0 = a_1$ and $b_0 = b_1$, and we are done. ■

The converse of this theorem is not true, as an example given in Part II of this paper will illustrate.

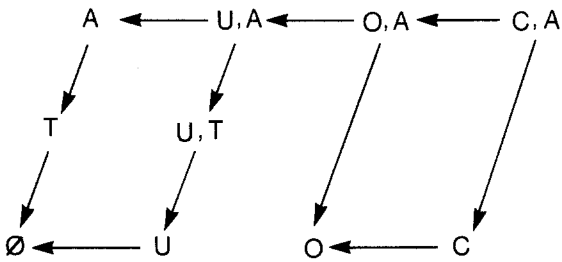

Fig. 1. Conditions on an ortho-algebra.

We have defined the following primary conditions on an ortho-algebra: transitivity (T), associativity (A), unique Mackey decomposition property (U), orthocoherence (O), and compatible coherence (C). We include Figure 1 to emphasize the connections between these conditions.

5. THE BOOLEAN STRUCTURE OF ORTHO-ALGEBRAS

In order to recover the Boolean block structure of an ortho-algebra, we first consider what conditions we must impose on an ortho-algebra so that the resulting structure is a Boolean algebra. Clearly, such an ortho-algebra must be associative. We will show that an associative ortho-algebra is an orthomodular partially ordered set if and only if it is orthocoherent, and is a Boolean algebra if and only if it is orthocoherent and pairwise compatible. These results derive from work by Foulis and Randäll.

We need the following lemmas:

Lemma 1. Let L be an associative ortho-algebra with $a, b \in L$ and $a' \perp b'$. If L is orthocoherent, then $a \wedge b$ exists and $a \wedge b = (a' \oplus b')'$.

Proof. Clearly, $a' \leq a' \oplus b'$, so $(a' \oplus b')' \leq a$. Similarly, $(a' \oplus b')' \leq b$, so $(a' \oplus b')'$ is a lower bound for a and b. Suppose $c \leq a, b$. Then $c \perp a'$ and $c \perp b'$ and, thus, by orthocoherence, $c \perp (a' \oplus b')$. Therefore, $c \leq (a' \oplus b')'$ and we are done. ■

Lemma 2. Let L be an associative ortho-algebra. If $a, b, c \in L$ with $a \perp b$ and $a \perp c$, then $b \leq c$ if and only if $(a \oplus b) \leq (a \oplus c)$.

Proof. Let $a, b, c \in L$ with $a \perp b$, $a \perp c$, and $b \leq c$. Then $b \perp c'$ and hence $b \perp (a \oplus (a \oplus c)')$. By associativity, $b \oplus (a \oplus (a \oplus c)') = (b \oplus a) \oplus (a \oplus c)'$, and thus $(b \oplus a) \perp (a \oplus c)'$ and we have $(b \oplus a) \leq (a \oplus c)$.

Conversely, let $(a \oplus b) \leq (a \oplus c)$. Then $(a \oplus c)' \leq (a \oplus b)'$, and by the first half of this proof, we have $a \oplus (a \oplus c)' \leq a \oplus (a \oplus b)'$. Therefore, $c' \leq b'$ and so $b \leq c$ as required. ■

Lemma 3. Let L be an associative ortho-algebra. Then (i) $a \perp b$ in L implies $a \oplus b$ is a minimal upper bound for a and b, and (ii) L is orthocoherent if and only if $a \perp b$ in L implies $a \vee b$ exists and $a \vee b = a \oplus b$.

Proof. (i) Let $a \perp b$ in L. Clearly, $a \oplus b$ is an upper bound for a and b. Suppose $a, b \leq k \leq a \oplus b$. Then there is an x in L with $k \oplus x = a \oplus b$. Therefore, $1 = (k \oplus x) \oplus (a \oplus b)' = k \oplus (x \oplus (a \oplus b)')$ and we have $k' = x \oplus (a \oplus b)'$. Thus $x \oplus (a \oplus b)' = k' \leq a' = b \oplus (a \oplus b)'$, and by Lemma 2, we have $x \leq b$. Similarly, $x \leq a$, and since $a \perp b$ we have $x = 0$ and we are done.

(ii) Let L be orthocoherent, and let $a \perp b$ in L. Let $c \in L$ with $a, b \leq c$. Then $a \perp c'$ and $b \perp c'$, and by orthocoherence, we have $(a \oplus b) \perp c'$ so $(a \oplus b) \leq c$ and we are done. Conversely, let $\{a, b, c\}$ be a pairwise orthogonal set. Then $a \leq c'$ and $b \leq c'$ so by assumption $a \oplus b \leq c'$ and we have $(a \oplus b) \perp c$. By associativity, then, $\{a, b, c\}$ is a jointly orthogonal set. By a similar argument and induction on the size of the set, it follows that L is orthocoherent. ■

We will use the following definition of an orthomodular partially ordered set, and include it here for reference:

Definition. Let $(P, \leq)$ be a partially ordered set with a map $'\colon P \to P$ defined on P, denoted $a \to a'$. Then P is called an *orthomodular partially ordered set* if it satisfies the following conditions for all $a, b \in P$: (i) if $a \leq b$, then $b' \leq a'$; (ii) $(a')' = a$; (iii) if $a \leq b$ then $a \vee (b \wedge a')$ exists and $b = a \vee (b \wedge a')$.

Theorem. Let L be an associative ortho-algebra. Then L is an orthomodular partially ordered set if and only if L is orthocoherent.

Proof. Let L be an orthomodular partially ordered set, and let $a \perp b$ in L. By Lemma 3, it suffices to show $a \vee b$ exists in L and $a \vee b = a \oplus b$. Since $a \perp b$, we have $a \leq b'$ and by (iii) of the above definition, we see $b' \wedge a'$ exists. Thus, $(b' \wedge a')' = a \vee b$ and $a \vee b$ exists, so $a \vee b \leq a \oplus b$. But by Lemma 3(i), we have $a \vee b = a \oplus b$ and we are done.

Conversely, let L be orthocoherent. Clearly L satisfies properties (i) and (ii) of the above definition, so it remains to show property (iii). Let $a \leq b$ in L. Then $a \perp b'$ and so $b = a \oplus (a \oplus b')' =$ (by Lemma 1) $a \oplus (a' \wedge b) =$ (by Lemma 3) $a \vee (a' \wedge b)$, and we are done. ■

Theorem. Let L be an associative ortho-algebra. Then L is a Boolean algebra if and only if L is orthocoherent and pairwise compatible.

Proof. Let L be an associative ortho-algebra which is also a Boolean algebra. By the preceding theorem, L is orthocoherent. Let $a, b \in L$. We must find a Mackey decomposition for a and b. Let $c = a \wedge b$. Then $c \leq a, b$ so we have $a = c \oplus (c \oplus a')'$ and $b = c \oplus (c \oplus b')'$. We must show $(c \oplus$

$a')' \perp (c \oplus b')'$. We have $(c \oplus a')' =$ (by Lemma 1) $a \wedge c' = a \wedge (a \wedge b)' = a \wedge (a' \vee b') =$ (since L is a Boolean algebra) $a \wedge b' \leq b' \leq (c \oplus b')$. Thus $(c \oplus a')' \perp (c \oplus b')'$ and, by orthocoherence, we have $\{c, (c \oplus a')', (c \oplus b')'\}$ is a Mackey decomposition for a, b.

Conversely, let L be orthocoherent and pairwise compatible. Then L is an orthomodular partially ordered set. Claim: L is a lattice. Let $a, b \in L$, and let $\{a_0, b_0, c\}$ be a Mackey decomposition for a, b. Let $a, b \leq x$. Then $a_0, b \leq x$, and by Lemma 3, $a_0 \oplus b \leq x$. Thus, $a \vee b$ exists, and is equal to $a_0 \oplus b_0 \oplus c$. Dually, $a \wedge b = (a' \vee b')'$ and L is a lattice. Claim: L is uniquely complemented. Let $a \vee b = 1$ and $a \wedge b = 0$. Let $\{a_0, b_0, c\}$ be a Mackey decomposition for a, b. Then $c \leq a, b$ so $c = 0$, and we see $a \perp b$. Thus $b = a'$ and we are done. ■

Definition. Let L be an ortho-algebra, and let A be a nonempty subset of L. Then A is said to be a *sub-ortho-algebra* of L if it satisfies the following conditions: (i) if $a \in A$, then $a' \in A$; (ii) if $a, b \in A$ with $a \perp b$, then $a \oplus b \in A$.

It is easy to see that if A is a sub-ortho-algebra of L, then $0, 1 \in A$ and A is itself an ortho-algebra under the inherited operations. As a corollary to the above theorem, we see that a sub-ortho-algebra A of an ortho-algebra L has the structure of a Boolean algebra if and only if A is associative, orthocoherent, and pairwise compatible.

Definition. Let L be an ortho-algebra, A a sub-ortho-algebra of L. Then A is called a *compatible sub-ortho-algebra* of L if A has the structure of a Boolean algebra under the inherited operations. A maximal compatible sub-ortho-algebra of L is called a (Boolean) *block* of L.

We wish to extend the definitions of jointly orthogonal and jointly compatible to include infinite subsets of an ortho-algebra L. First, we need the following theorem:

Theorem. Let A be a finite subset of an ortho-albegra L. Then (1) A is jointly orthogonal if and only if it is contained in a compatible sub-ortho-algebra and is pairwise orthogonal; (2) A is jointly compatible if and only if it is contained in a compatible sub-ortho-algebra.

Proof. (1) Let A be jointly orthogonal. Define $A^+ = A \cup \{(\sum A)'\}$. Using property (vi) in the definition of an ortho-algebra, it can be shown that A^+ is jointly orthogonal. Let $S = \{\sum A_0 \colon A_0 \subseteq A^+\}$. By the definition of jointly orthogonal, S is associative and orthocoherent. Let $a, b \in S$, say, $a = \sum A_1$ and $b = \sum A_2$ with $A_1, A_2 \subseteq A^+$. It can be shown that $\{\sum (A_1 - A_2), \sum (A_2 - A_1), \sum (A_1 \cap A_2)\}$ is a Mackey decomposition for a, b in S, and so S is pairwise compatible. Therefore, S is a compatible sub-ortho-algebra, and we are done. The converse is easily seen to be true.

(2) This is a direct corollary to 1. ■

Now, for an arbitrary subset A of an ortho-algebra L, we say A is *jointly compatible* if it is contained in a compatible sub-ortho-algebra, and we say A is *jointly orthogonal* if it is jointly compatible and pairwise orthogonal.

6. EQUIVALENCE OF ORTHO-ALGEBRAS AND BOOLEAN ATLASES

Theorem. (1) Every Boolean atlas defines an ortho-algebra in a natural way.

(2) Every ortho-algebra defines a Boolean atlas in a natural way.

Proof. (1) Let $\mathcal{B} = (B_i\colon i \in I)$ be a Boolean atlas. We define an ortho-algebra as follows: $L = \bigcup B_i$, $0 = 0_i$ and $1 = 1_i$ for any $i \in I$, $a' = C_i(a)$ for any i such that $a \in B_i$, we say $a \perp b$ if there is an $i \in I$ with $a, b \in B_i$ and $M_i(a, b) = 0$, and, if $a \perp b$, $a \oplus b$ is defined to be $J_i(a, b)$ for any i with $a, b \in B_i$. It is clear that these definitions are well defined, and it can be shown that the properties of an ortho-algebra are satisfied.

(2) Let L be an ortho-algebra. Let $(B_i\colon i \in I)$ be the set of blocks on L. It is clear that properties (i)–(iv) in the definition of a Boolean atlas are satisfied by this set. To see property (v), let $a, b \in B_i$ with $M_i(a, b) = 0$. B_i is pairwise compatible, so there is a Mackey decomposition $\{a_0, b_0, c\}$ for a, b in B_i. But then $c \leq a, b$ and so $c = 0$ and we have $a \perp b$. Thus, by Lemma 3 and orthocoherence of B_i, we have $J_i(a, b) = a \oplus b$, and we are done. ■

Note that the conditions for a sub-ortho-algebra to be a compatible sub-ortho-algebra are finite in nature. Therefore, by Zorn's lemma, we see that every compatible sub-ortho-algebra is contained in a block. Furthermore, for every element a in an ortho-algebra L, the set $\{0, a, a', 1\}$ is a compatible subortho-algebra of L, so every element of L is contained in at least one block. Hence the set of elements of the Boolean atlas associated with an ortho-algebra L is identical to the original elements in L. Furthermore, if L is an ortho-algebra and we construct the corresponding Boolean atlas, and then, from this, construct the corresponding ortho-algebra as described above, this ortho-algebra will be identical to the original one. If, however, we start with a Boolean atlas, construct the corresponding ortho-algebra and then the corresponding Boolean atlas, the resulting atlas may be larger. Whereas every one of the original Boolean algebras is a block, there may be blocks that are not among the original Boolean algebras. This procedure of obtaining a (perhaps) larger Boolean atlas from an initial Boolean atlas $\mathcal{B}$ is called *filling out* $\mathcal{B}$. If the resulting atlas is identical to the original one, we say that $\mathcal{B}$ is *full.* Notice that the atlas of blocks on an ortho-algebra is always full.

It can be shown that our definitions of orthogonal, compatible, jointly orthogonal, and jointly compatible on an ortho-algebra carry over to their namesakes on a Boolean atlas, and vice-versa. Hence, our definitions of orthocoherence and compatible coherence on ortho-algebras may be extended to the equivalent notions on Boolean atlases.

In a Boolean atlas, it is clear that if two elements are not in a common block, their meet and join may not exist. In addition, however, even if we have $a, b \in B_i$ with $a \perp b$, the local supremum $(a \oplus b)$ need not be a global supremum. An example illustrating this will be included in Part II. In fact, it is easily seen that $a \oplus b$ being a global supremum for every orthogonal pair a, b in an ortho-algebra is equivalent to orthocoherence. It is not hard to show that the associated Boolean atlas to an ortho-algebra L is a Boolean manifold if and only if L satisfies the unique Mackey decomposition property.

7. SUMMARY

Of the previous work done in the area of quantum logics, we see that the most general system is that of Foulis and Randall. The logic associated with a manual, called an operational logic (OL), will be shown in Part II of this paper to be an associative ortho-algebra. The quantum mechanical systems discussed by Domotar are equivalent to Boolean manifolds (BM), or ortho-algebras which satisfy the unique Mackey decomposition property. We showed earlier that orthomodular partially ordered sets (OMP) are orthocoherent associative ortho-algebras, or, equivalently, full Boolean atlases in which every orthogonal pair has a global supremum. Finally, partial Boolean algebras (PBA), as defined by Kochen and Specker, are Boolean manifolds which are compatible coherent, or, equivalently, compatibly coherent ortho-algebras. Gudder (1972) had shown previously, as we see here, that an orthomodular partially ordered set is a partial Boolean algebra if and only if it is compatibly coherent, and that a partial Boolean algebra is an orthomodular partially ordered set if and only if it is transitive. Thus, the common system is the class of transitive partial Boolean algebras, or, equivalently, the class of compatibly coherent orthomodular partially ordered sets, or, as we now see, compatibly coherent associative ortho-algebras.

Figure 2 corresponds directly to Figure 1. In it, we have labeled the nodes with the appropriate system, and the edges with the condition(s) required to obtain that system. The major quantum logical systems, as just described, are in boxes. The intermediate systems included are transitive ortho-algebras (TOA), transitive Boolean manifolds (TBM), associative Boolean manifolds (ABM), orthocoherent ortho-algebras (OOA), and com-

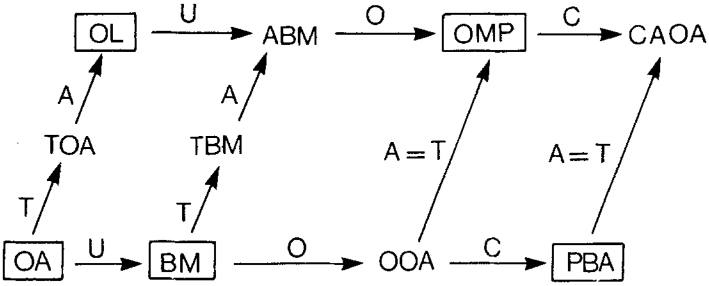

Fig. 2. Quantum logical systems.

patibly coherent associative ortho-algebras (CAOA). The conditions are labeled as in Figure 1.

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